Estimating the Probability Distribution of von Mises Stress for Structures Undergoing Random Excitation, Part 1: Derivation

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Abstract

The primary purpose of finite element stress analysis is to estimate the reliability of engineering designs. In structural applications, the von Mises stress due to a given load is often used as the metric for evaluating design margins. For deterministic loads, both static and dynamic, the calculation of von Mises stress is straightforward [1]. For random load environments typically defined in terms of power spectral densities, the deterministic theory normally applied to compute acceleration, displacement, or stress tensor responses cannot be applied directly to calculate the probability distribution of von Mises stress, a nonlinear function of the linear stress components. The probability distribution of von Mises stress is not Gaussian, nor is it centered about zero as are the stress components. Therefore, the form of the von Mises probability distribution must be determined and the parameters of that distribution must be found. In a previous paper [2] the authors presented a computationally efficient method of estimating the RMS value of von Mises stress for the case of input force of Gaussian distribution with zero mean. Here we present a procedure for estimating the full probability distribution for such cases.

The reliability calculations for a structure of ductile material require a linear model for the structure and a statistical specification of the input forces. In principle, from the linear

Nomenclature

N_ω	number of frequencies in PSD of force specification
$F(\omega, T)$	FFT of imposed load sampled over period T
$E[\ ]$	expected value operator
$S_F(\omega)$	cross-spectral density matrix of imposed loads
$q^n(t)$	modal coordinate of n'th mode
$q(t)$	array of all modal coordinates
$\hat{q}(\omega, T)$	Fourier transform of $q(t)$ sampled over period T
$\sigma(t, x)$	stress vector at location x and time t
$\sigma_n(x)$	stress vector at location x associated with mode n
$p(t, x)$	von Mises stress at location x and time t
$H(\omega)$	transfer function from modal forces to modal coordinates
S_q	covariance matrix of modal coordinates
C	covariance matrix defined in Eq. 16
D^2	diagonal intrinsic covariance matrix defined in Eq 17
N	rank of D
$E(\{D\}, Y)$	N dimensional ellipse about origin whose semi axes are the diagonals of D
$V_U(\{D\}, Y, \alpha)$	collection of N dimensional boxes that contain the ellipse $E(\{D\}, Y)$, indexed by parameter α.
$V_L(\{D\}, Y, \alpha)$	collection of N dimensional boxes that are contained in the ellipse $E(\{D\}, Y)$, indexed by parameter α.

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model one can deduce all required transfer functions. Input forces are specified by their auto spectral densities with all phase content missing. In the case of multiple force inputs, the forces may be specified by a cross spectral density matrix over frequency. It is demonstrated here how that information can be used to calculate the probability distributions for the von Mises stress at different locations on the body. An integral formulation is presented for cumulative probabilities and a method for approximating those integrals is also presented.

A natural alternative to the method presented here would involve computing long series of values of von Mises stress and trying to deduce probability distributions from histograms of that data. Indeed this method was used to check and compare results generated by the core method of this paper. One notes that there are three serious deficiencies of this alternative, time series approach:

- this process is of order $N_\omega^2 \log N_\omega$ for each output location. This is prohibitively expensive for large models.
- from the numerical data, one has no systematic method of deducing the asymptotic properties of the distribution. It is these asymptotic properties that are important in reliability estimation.
- in creating these Fourier components of stress from the available force data, one would have to postulate (invent) phase information for the input forces.

Derivation

Where the applied load involves either forces applied at several locations or forces applied at one location but in more than one direction, the loads are usually represented by the cross spectral density matrix:[3,4]

$$S_F(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} E[\bar{F}(\omega, T) F(\omega, T)^T] \qquad \text{Eq 1}$$

where $F(\omega, T)$ is the Fourier transform of the vector of force components sampled over at period T; $(\ )^T$ denotes the matrix transpose; $(\bar{\ })$ denotes the complex conjugate; and $E[\ ]$ is the expected value operator (estimates for which are obtained by ensemble averaging.) In the case of a single scalar input force, this reduces to the auto spectral density. The above assumes that the load constitute continuous processes.

The stress at the point in question can be assembled from the contributions of each mode:

$$\sigma(t, x) = \sum_n q^n(t) \sigma_n(x) \qquad \text{Eq 2}$$

where q^n is the nth modal coordinate and $\sigma_n(x)$ is the stress vector at location x associated with that mode. (The stress vector contains the six non-redundant terms in the stress tensor.)

The square of the von Mises stress can be expressed as a quadratic operator on the stress vector

$$p^2(t, x) = \sigma(t, x)^T A\ \sigma(t, x) \qquad \text{Eq 3}$$

where

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & & & \\ -\frac{1}{2} & 1 & -\frac{1}{2} & & & \\ -\frac{1}{2} & -\frac{1}{2} & 1 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix} \qquad \text{Eq 4}$$

It is clear that to obtain the probability distribution of von Mises stress we must obtain an expression involving the vector $q(t)$ of components $q_n(t)$ in terms of what is known about the structure and the applied loads. This is done as follows.

Because we assume that we have a linear model for the structure, we have the eigen modes, which can be used to map the force information into modal force components:[6]

$$\hat{f}(\omega, T) = \Phi^T F(\omega, T) \qquad \text{Eq 5}$$

where Φ is the matrix whose columns are the eigen modes of the system. The transfer functions are also known, in particular the mapping from the Fourier transform of modal forces to the Fourier transform $\hat{q}(\omega, T)$ of the vector of modal coordinates is diagonal matrix $H(\omega)$ [6]:

$$\hat{q}(\omega, T) = H(\omega) \hat{f}(\omega, T) = H(\omega) \Phi^T F(\omega, T) \qquad \text{Eq 6}$$

This is a frequency space expression for modal response in terms of the applied load. We now need to express the structural response in terms of what is actually known about the applied loads, and that is statistical in nature. We take an outer expansion of Equation 6 with itself and take the expected value:

$$\begin{aligned} E[\overline{\hat{q}(\omega)} \hat{q}(\omega)^T] &= \lim_{T \to \infty} \frac{1}{2\pi T} E[\overline{\hat{q}(\omega, T)} \hat{q}(\omega, T)^T] \\ &= \overline{H(\omega)} \Phi^T S_F(\omega) \Phi H(\omega) \end{aligned} \qquad \text{Eq 7}$$

The above is mapped from frequency space to time space with aid of Parsevals theorem[7] to yield the covariance matrix S_q at zero lag:

$$S_q = E[q(t) q(t)^T] = Re\left\{ \int_0^\infty E[\overline{\hat{q}(\omega)} \hat{q}(\omega)^T] d\omega \right\} \qquad \text{Eq 8}$$

We use the standard methods to decompose S_q and to map the modal coordinates into uncorrelated variables. Observing that S_q is symmetric and positive semi-definite, its singular value decomposition is [5]

$$S_q = Q X^2 Q^T \qquad \text{Eq 9}$$

where X is a diagonal matrix whose dimension is the rank of S_α and Q is a rectangular matrix having the property that $Q^T Q = I$ is the identity whose dimension is the rank of S. (Here we have retained only the nonzero terms of the diagonal matrix and the corresponding columns of the rotation matrix. For a symmetric, positive semi-definite matrix eigen analysis and singular value decomposition are the same.) Defining

$$\beta = X^{-1} Q^T q , \quad \text{Eq 10}$$

we find that components of β are independent, identically distributed (IID) Gaussian processes, each with unit variance(GUV):

$$E[\beta\beta^T] = I \quad \text{Eq 11}$$

We define another set of random variables by

$$q' = QX\beta = QQ^T q . \quad \text{Eq 12}$$

A little algebra shows that

$$E[(q-q')(q-q')^T] = 0 \quad \text{Eq 13}$$

from which we conclude that

$$q = q' = QX\beta \quad \text{Eq 14}$$

In our new coordinates, β, the square of the von Mises stress is

$$p^2 = \beta^T C \beta \quad \text{Eq 15}$$

where

$$C_{kl} = (\sigma_m^T A \sigma_n) K_{mk} K_{nl} \quad \text{Eq 16}$$

and $K = QX$. Matrix C is square having dimensionality equal to the rank of S_q but possibly much lower rank. Because the rank of A is five, the rank of C can be at most five. Further, the rank of C is also bounded by the rank of S_q, and by the dimensionality of the stress vectors.

We exploit the symmetry and the positive semi-definiteness of C in doing its singular value decomposition:

$$C = RD^2R^T \quad \text{Eq 17}$$

where the matrix D is diagonal and has dimension equal to the rank of C and R is a rectangular matrix having property that $R^T R = I$ is the identity matrix whose dimension is the rank of C. The von Mises stress is now

$$p^2 = \beta^T R D^2 R^T \beta \quad \text{Eq 18}$$

This suggests yet another change of variables:

$$y = R^T \beta \quad \text{Eq 19}$$

It is easily shown that the elements of y are IID, GUV. The advantages of the above transformation are first that it reduces the number of random variables of this problem to the rank of C (at most five) and second that it aligns the random variables in the directions of the axes of the ellipsoids of constant von Mises stress.

$$p^2 = y^T D^2 y = \sum_n y_n^2 D_n^2 \quad \text{Eq 20}$$

The average value of the square of the von Mises stress is

$$\begin{aligned} E[p^2] &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p^2 \prod \rho_r(y_r) \prod dy_r \\ &= \int_{-\infty} \dots \int_{-\infty} y^T D^2 y \prod \rho_r(y_r) \prod dy_r \end{aligned} \quad \text{Eq 21}$$

Noting that $\int_{-\infty}^{\infty} y_r^2 \rho_r(y_r) dy_r = 1$, the above becomes

$$E[p^2] = \sum_r D_r^2 \quad \text{Eq 22}$$

We see that D_r^2 is the contribution of the r 'th random process to $E[p^2]$ and the rank of D is the number of independent random processes taking place at that location. For convenience, we refer to $\sqrt{E[p^2]}$ as $\bar{p}$.

We now calculate the probability of the von Mises stress being less than some value Y:

$$P(p < Y) = \int_{E(\{D\}, Y)} \prod \rho_r(y_r) \prod dy_r \quad \text{Eq 23}$$

where $E(\{D\}, Y)$ is the N-dimensional ellipsoid containing points y associated with von Mises stress less than Y:

$$E(\{D\}, Y) = \{y : (y^T D^2 y) \le Y\} \quad \text{Eq 24}$$

and N is the rank of D. The integral of Equation 23 is difficult to evaluate.

Quadrature by Boxes

We discuss here how to achieve upper and lower bounds for the integral in Equation 23. This discussion then leads to reasonably good approximations for that integral.

We first note that the integral of $\prod \rho_r(y_r) dy_r$ over an N dimensional box, B, having faces normal to each of the coordinates y_r, can be calculated analytically:

$$P_B = \int_B \prod_{k=1}^{N} \rho_r(y_r) dy_r$$

$$= \prod_{k=1}^{N} [\Phi(y_{r,max}) - \Phi(-y_{r,min})] \quad \text{Eq 25}$$

where $y_{r,max}$ and $y_{r,min}$ define the boundaries of B and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-s^2/2) ds \quad \text{Eq 26}$$

is the cumulative distribution for a Gaussian distribution[9]. The quantity $\Phi(x) - \Phi(-x)$ is the normal probability integral[8].

We next consider volumes $V_L(\{D\}, Y, \alpha)$ and $V_U(\{D\}, Y, \alpha)$ each of which is a union of N dimensional boxes selected so that

$$V_L(\{D\}, Y, \alpha) \subseteq E(\{D\}, Y) \subseteq V_U(\{D\}, Y, \alpha) \quad \text{Eq 27}$$

The parameter α is an indicator of the level of refinement so that $V_L(\{D\}, Y, \alpha), V_U(\{D\}, Y, \alpha) \to E(\{D\}, Y)$ as $\alpha \to \infty$.

These contained and containing volumes are illustrated for a problem of two processes ($N = 2$) in the figure below.

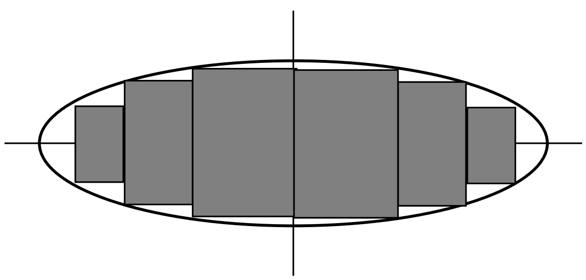

Figure 1. A collection of boxes entirely contained in the ellipsoid, is an admissible $V_L(\{D\}, Y, \alpha)$

Expressing each of these volumes in terms of its component boxes:

$$V_L(\{D\}, Y, \alpha) = \bigcup_k B_{L,k}(\{D\}, Y, \alpha) \quad \text{Eq 28}$$

and

$$V_U(\{D\}, Y, \alpha) = \bigcup_k B_{U,k}(\{D\}, Y, \alpha) \quad \text{Eq 29}$$

The integral is now approximated by:

$$\int_{V_L(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r = \int_{\bigcup_k B_{L,k}(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r$$

$$= \sum_k P_{B_{L,k}(\{D\}, Y, \alpha)} \quad \text{Eq 30}$$

and by

$$\int_{V_U(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r = \int_{\bigcup_k B_{U,k}(\{D\}, Y, \alpha)} \prod \rho_r(y_r) dy_r$$

$$= \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \quad \text{Eq 31}$$

Recalling Equation 27 and observing that the integrand is positive, we have upper and lower bounds for $P(p < Y)$:

$$\sum_k P_{B_{L,k}(\{D\}, Y, \alpha)} \le P(p < Y) = \int_{E(\{D\}, Y)} \prod \rho_r(y_r) \prod dy_r$$

$$\le \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \quad \text{Eq 32}$$

We also note that

$$\sum_k P_{B_{L,k}(\{D\}, Y, \alpha)}, \sum_k P_{B_{U,k}(\{D\}, Y, \alpha)} \to P(p < Y) \quad \text{Eq 33}$$

as $\alpha \to \infty$ and that convergence is assessed by the difference of the upper and lower bound quadrature.

The mathematics discussed above has been implemented in a simple recursive C language procedure which is listed in the Appendix.

Numerical Comparison

Following is a plot of the exact and the approximate calculations for the case of two independent processes each contributing equally to von Mises stress ($D_1 = D_2 = 1$). In this case the probability density is a Rayleigh distribution. This case

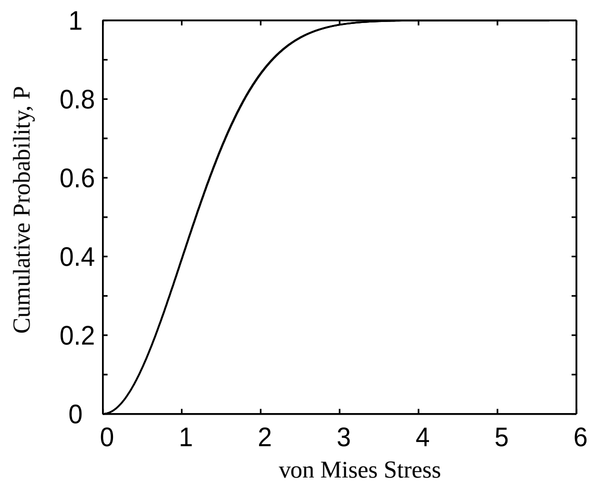

Figure 2. Comparison of exact ($[1 - \exp(-Y^2/2)]$**) cumulative distribution function for** $D_1 = D_2 = 1$ **and numerical quadrature. Quadrature generates upper and lower bounds which almost overly the analytic curve.**

is one for which a simple closed-form expression can be achieved for the exact integral.

The numerical quadrature used here employed 128^2 boxes in the calculation of the lower bound and 129^2 in the calculation of the upper bound. The error is shown in the following figure. The

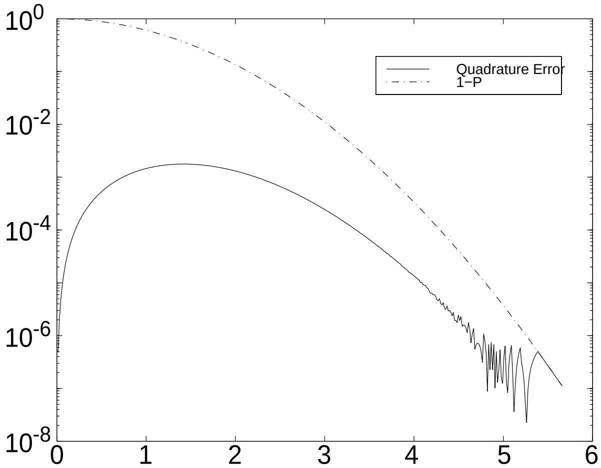

Figure 3. The quadrature error and $1 - P$ for the cumulative distribution function for $D_1 = D_2 = 1$.

maximum error in this case was $2.0x10^{-3}$ and occurred near the RMS value of von Mises stress. In the quadrature employed, the magnitude of the upper-bound error was almost exactly the magnitude of the lower-bound error. Also interesting is the comparison of the magnitude of the error and the function $1 - P$, the difference between the cumulative probability and 1.0. It is seen that the error stays substantially below $1 - P$, indicating that the quadrature remains accurate even out to high values of von Mises stress. In other cases, comparison could be made only between the upper-bound and lower-bound quadratures.

Summary

The authors have derived and presented an expression for the cumulative probability distribution for the von Mises stress resulting from random loadings that are Gaussian and of zero mean. This is an important result for reliability of structures subject to such loads.

Additionally, a convenient set of expressions were derived for upper and lower bounds to the cumulative probability.

Finally, it should be noted that the derivation of the cumulative probability integral and of the approximations for it could also be applied to any other quadratic function of the load.

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Appendix: Code Fragment for Recursive Calculation of Lower-Bound Quadrature

```
// recursive routine to calculate a lower
//   bound for the integral
double root2 = sqrt(2.0);
double slabL(double *D,     int generation,
             double remain, double *xi,
             int Inner)
{
   double ymax=sqrt(remain)/D[generation];
   if(generation==4)
      return(erf(ymax/root2));
   if(D[generation+1] < D[0]*0.01)
      return(erf(ymax/root2));
   double sum=0;
   double y1, y2;
   y1 = 0;
   int i;
   // in the following, it is assumed that
   //xi[Inner] < 1;
   for(i=0; i<Inner; i++){
      y1 = xi[i]  *ymax;
      y2 = xi[i+1]*ymax;
      double remain2 = remain-
         (y2*D[generation])
         *(y2*D[generation]);
      sum += (erf(y2/root2) -
         erf(y1/root2))*
         slabL( D,     generation+1,
           remain2, xi, Inner);
  }
  return(sum);
}
```